Answer Key

NAME:

Math 150 Exam 3

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. True or False?

a) If f and g are continuous on [a, b], then

$$\int_a^b [f(x)g(x)]dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$$ [2 pts]

False

b) $\int_{-1}^{1}\left(x^5 - 6x^9 + \frac{\sin x}{(1+x^4)^2}\right)dx = 0$. [2 pts]

True

c) All continuous functions have antiderivatives. [2 pts]

True

d) If $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for $0 \le x \le 1$ [2 pts]

False

e) $\int_0^3 e^{x^2}dx = \int_0^5 e^{x^2}dx + \int_5^3 e^{x^2}dx$ [2 pts]

True

2. Evaluate $\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}x}\right)dx$ [10 pts]

$$\int \frac{d}{dx}\left(e^{\tan^{-1}x}\right)dx = e^{\tan^{-1}x}$$

so $$\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}x}\right)dx = e^{\tan^{-1}x}\Big|_0^1 = e^{\tan^{-1}1} - e^{\tan^{-1}0}$$

$$= \boxed{e^{\frac{\pi}{4}} - 1}$$

3. Evaluate $\int_0^3 |x^2 - 4| dx$ [10 pts]

$x^2 - 4 < 0$ if $x < 2$ so

$$\int_0^3 |x^2-4|dx = \int_0^2 -(x^2-4)\,dx + \int_2^3 (x^2-4)\,dx$$

$$= 8 - \frac{8}{3} - 4 + \frac{27}{3} - \frac{8}{3} = 4 + \frac{27}{3} - \frac{16}{3} = 4 + \frac{11}{3}$$

$$= \frac{12+11}{3} = \boxed{\frac{23}{3}}$$

4. Evaluate $\int_0^1 v^2 \cos(v^3) dv$ [10 pts]

let $u = v^3$; $du = 3v^2 dv$; $\frac{1}{3} du = v^2 dv$

so $$\int_0^1 v^2 \cos(v^3) dv = \int_{v=0}^{v=1} v^2 \cos(v^3) dv = \int_0^1 \frac{1}{3} \cos u \, du$$

$$= \frac{1}{3} \sin u \Big|_{u=0}^{u=1} = \boxed{\frac{1}{3} \sin(1)}.$$

5. Evaluate $\int_{-1}^{1} \frac{\sin x}{1+x^2}\,dx$ [10 pts]

$f(x) = \frac{\sin x}{1+x^2}$. is odd: $f(-x) = \frac{\sin(-x)}{1+(-x)^2} = -\frac{\sin x}{1+x^2}$

so $\int_a^a f(x)\,dx = 0$. In particular

$$\boxed{\int_{-1}^{1} f(x)\,dx = 0}$$

6. Find the derivative of the function $f(x) = \int_{\sqrt{x}}^{x} \frac{e^t}{t}\,dt$ [10 pts]

$$f'(x) = \frac{e^x}{x} - \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^x}{x} - \frac{e^{\sqrt{x}}}{2x}.$$

This calculation follows from the fact that

$$\int_{\sqrt{x}}^{x} \frac{e^t}{t}\,dt = \int_{\sqrt{x}}^{0} \frac{e^t}{t}\,dt + \int_0^x \frac{e^t}{t}\,dt = \int_0^x \frac{e^t}{t}\,dt - \int_0^{\sqrt{x}} \frac{e^t}{t}\,dt.$$

7. Express the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{4}{n}\left(2+\frac{4}{n}\right)\ln\left(1+\left(2+\frac{4}{n}\right)^2\right)$ as a definite integral. Do not evaluate. [10 pts]

$a=2 \quad \Delta x_n = \frac{b-a}{n} = \frac{4}{n}$ Hence $b = 2+4=6$

The integral is therefore $\boxed{\int_2^6 x\ln(1+x^2)\,dx}$

8. Use the properties of integrals to verify the inequality $\int_0^1 x^2 \sin\sqrt{x}\,dx \le \frac{1}{3}$ [10 pts]

$$\int_0^1 x^2 \sin\sqrt{x}\,dx \leq \int_0^1 x^2\,dx = \frac{1}{3}$$

The above inequality holds, because $\sin\sqrt{x} \leq 1$ on $[0,1]$, which implies $x^2\sin\sqrt{x} \leqslant x^2$.

9. Let $f(x) = 2x + x^2$. Use the right-hand method to compute the area $\int_{-1}^{1} f(x)dx$. [Hint: to compute R_n, you will need the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$] [10 pts]

$R_n = \Delta x_n \sum_{k=1}^{n} f(a+\Delta x_n \cdot k)$ where $a=-1$ and $\Delta x_n = \frac{b-a}{n} =$

$= \frac{1-(-1)}{n} = \frac{2}{n}$. Thus

$$R_n = \frac{2}{n} \sum_{k=1}^{n} f\left(-1+\frac{2}{n}k\right) = \frac{2}{n} \sum_{k=1}^{n} \left[2\left(-1+\frac{2}{n}k\right) + \left(-1+\frac{2}{n}k\right)^2\right]$$

$$= \frac{2}{n}\left(\sum_{k=1}^{n} (-2) + \cancel{\frac{4}{n}\sum_{k=1}^{n} k} + \sum_{k=1}^{n} (-1)^2 + \cancel{\frac{(-4)}{n}\sum_{k=1}^{n} k} + \frac{4}{n^2}\sum_{k=1}^{n} k^2 \right)$$

$$= \frac{2}{n}\left(-2n + n + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right)$$

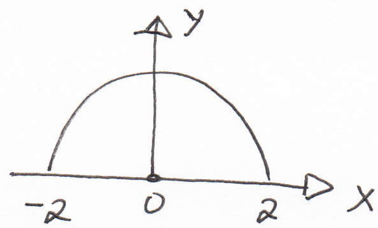
$$= \frac{2}{n}\left(-n + \frac{2(n+1)(2n+1)}{3n}\right) = \frac{-2n}{n} + \frac{4(n+1)(2n+1)}{3n^2}$$

Thus $\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left(-2 + \frac{4(n+1)(2n+1)}{3n^2}\right) = -2 + \frac{8}{3} = \boxed{\frac{2}{3}}$

10. Evaluate $\int_{-2}^{2}\sqrt{4-x^2}\,dx$ [10 pts]

This integral represents the upper portion of the circle

$x^2+y^2=4$



Therefore $\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{1}{2}\pi\cdot 2^2 = \frac{2\pi}{1} = \boxed{2\pi}$

Extra-Credit

11. Evaluate $\lim_{n\to\infty}\frac{\pi}{2n}\left[Cos\left(\frac{\pi}{2n}\right)+Cos\left(\frac{2\pi}{2n}\right)+Cos\left(\frac{3\pi}{2n}\right)+\ldots+Cos\left(\frac{n\pi}{2n}\right)\right]$

[10 pts]

The limit is a Riemann sum

$$\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n}Cos\left(\frac{k\pi}{2n}\right) =$$

$= \int_0^{\frac{\pi}{2}} Cos(x)\,dx$ where the bounds of integration were obtained by noting that $\Delta x_n = \frac{b-a}{n} = \frac{\pi/2}{n} = \frac{\pi}{2n}$

$a = \left.\frac{k\pi}{2n}\right|_{k=0} = 0$, $b = \left.\frac{k\pi}{2n}\right|_{k=n} = \frac{\pi}{2}$.

Thus the limit is $\int_0^{\frac{\pi}{2}} Cos(x)\,dx = Sin\frac{\pi}{2} = \boxed{1}$